Solutions: Homework 3

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October 21, 2019

Problem 1. We say that $f : X \to \mathbb{C}$ is bounded if there is a constant $M > 0$ with $|f(x)| \leq M$ for all x in X. Show that if f and g are bounded uniformly continuous (Lipschitz) functions from X into $\mathbb{C}$ then so is fg.

Proof. Let d denote the metric on X. Since f and g are bounded, there exists $M > 0$ such that $|f(x)| \leq M$ and $|g(x)| \leq M$ for all x in X. So, $|(fg)(x)| \leq M^2$ for all x in X and hence fg is bounded. Now, let $\epsilon > 0$. By the uniform continuity of f and g, there exists $\delta > 0$ such that $|f(x) - f(y)| < \epsilon/2M$ and $|g(x) - g(y)| < \epsilon/2M$ for all x, y in X such that $d(x, y) < \delta$. Then, for any x, y in X such that $d(x, y) < \delta$, we have

$$|f(x)g(x) - f(y)g(y)| = |f(x)g(x) - f(x)g(y) + f(x)g(y) - f(y)g(y)|$$

$$\leq |f(x)||g(x) - g(y)| + |g(y)||f(x) - f(y)| \leq M.\frac{\epsilon}{2M} + M.\frac{\epsilon}{2M} = \epsilon$$

This proves the uniform continuity of fg.
Now, let f and g be bounded (with bound M) Lipschitz functions with constant M'. Then $|f(x) - f(y)| \leq M'd(x, y)$ and $|g(x) - g(y)| \leq M'd(x, y)$ for all $x, y \in X$. Then, as above,

$$|f(x)g(x) - f(y)g(y)| \leq |f(x)||g(x) - g(y)| + |g(y)||f(x) - f(y)|$$

$$\leq MM'd(x, y) + MM'd(x, y) = 2MM'd(x, y)$$

So, fg is Lipschitz with constant $2MM'$.

□

Problem 2. Suppose $f : X \to \Omega$ is uniformly continuous; show that if $\{x_n\}$ is a Cauchy sequence in X then $\{f(x_n)\}$ is a Cauchy sequence in Ω. Is this still true if we only assume that f is continuous?

Proof. Let d denote the metric on X and let ρ denote the metric on Ω. Let $\epsilon > 0$. Then, by the uniform continuity of f, there exists $\delta > 0$ such that $\rho(f(x), f(y)) < \epsilon$ whenever $d(x, y) < \delta$. By the Cauchy-ness of $\{x_n\}$, there exists $N \in \mathbb{N}$ such that $d(x_n, x_m) < \delta$ for all $n, m \geq N$. This implies that $\rho(f(x_n), f(x_m)) < \epsilon$ for all $n, m \geq N$. As $\epsilon > 0$ was arbitrary, we conclude that $\{f(x_n)\}$ is Cauchy in Ω.
This is not true if f is just assumed to be continuous. For example, take $f : (0, 1) \to (1, \infty)$ given by $f(x) = 1/x$. Then the sequence $\{1/n\}$ is Cauchy in $(0, 1)$ but $\{f(1/n)\}$ is not Cauchy in $(1, \infty)$. □

Problem 3. Suppose that Ω is a complete metric space and that $f : (D, d) \to (\Omega; \rho)$ is uniformly continuous, where D is dense in (X, d). Use Problem 2 to show that there is a uniformly continuous function $g : X \to \Omega$ with $g(x) = f(x)$ for every x in D.

Proof. Let x in X. We can then choose a sequence $\{x_n\}$ in D that converges to x in X. Since $\{x_n\}$ is a Cauchy sequence (because it is convergent), by Problem 2, we know that $\{f(x_n)\}$ is a Cauchy sequence in Ω. Since Ω is complete, it converges in Ω to a limit, which we shall denote by $g(x)$. Now, let $\{y_n\}$ be another sequence in D converging to x in X. Then it is easy to see that the sequence $x_1, y_1, x_2, y_2, ...$ is a Cauchy sequence in D converging to x in X. So, the sequence $f(x_1), f(y_1), f(x_2), f(y_2), ...$ in Ω is Cauchy and has a convergent subsequence $\{f(x_n)\}$ converging to $g(x)$. This implies that the subsequence $\{f(y_n)\}$ also converges to $g(x)$. So, $g(x)$ is an element of Ω that is dependent only on x and not on the choice of sequence in D. So the function $g : X \to \Omega$ is well-defined. Clearly, if $x \in D$, choosing the sequence $\{x_n = x\}$ in D implies that $g(x) = f(x)$. Now, let $\epsilon > 0$. Since f is uniformly continuous, there exists $\delta > 0$ such that $\rho(f(x), f(y)) < \epsilon$ whenever x, y in D with $d(x, y) < \delta$. Let x, y in X be such that $d(x, y) = \delta - r$ with $0 < r \leq \delta$. Choose 2 sequences $\{x_n\}$ and $\{y_n\}$ in D converging to x and y respectively. Choose N large enough such that $d(x_N, x) < r/2, d(y_N, y) < r/2, \rho(f(x_N), g(x)) < \epsilon$ and $\rho(f(y_N), g(y)) < \epsilon$. Then

$$d(x_N, y_N) \leq d(x_N, x) + d(x, y) + d(y, y_N) < r/2 + \delta - r + r/2 = \delta.$$

This implies that $\rho(f(x_N, y_N)) < \epsilon/3$. So,

$$\rho(g(x), g(y)) \leq \rho(g(x), f(x_N)) + \rho(f(x_N), f(y_N)) + \rho(f(y_N), g(y)) < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon.$$

This shows that g is uniformly continuous. □

Problem 4. Let $\{f_n\}$ be a sequence of uniformly continuous functions from (X, d) into (Ω, p) and suppose that $f = u\text{-}\lim f_n$ exists. Prove that f is uniformly continuous. If each f_n is a Lipschitz function with constant M_n and $\sup M_n < \infty$, show that f is a Lipschitz function. If $\sup M_n = \infty$, show that f may fail to be Lipschitz.

Proof. Let $\epsilon > 0$. Fix N large enough such that $p(f_N(x), f(x)) < \epsilon/3$ for all x in X. Since f_N is uniformly continuous, there exists $\delta > 0$ such that $p(f_N(x), f_N(y)) < \epsilon/3$ for all x, y in X with $d(x, y) < \delta$. Then, for all x, y in X with $d(x, y) < \delta$, we have

$$p(f(x), f(y)) \leq p(f(x), f_N(x)) + p(f_N(x), f_N(y)) + p(f_N(y), f(y)) < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon.$$

So f is uniformly continuous.
Now suppose the f_n's are Lipschitz functions with constant M_n. So $p(f_n(x), f_n(y)) \leq M_n d(x, y)$ for all x, y in X. Let $M = \sup M_n$. Pick N large enough such that $p(f(x), f_N(x)) < \epsilon/2$ for all x in X. Then, we have

$$p(f(x), f(y)) \leq p(f(x), f_N(x)) + p(f_N(x), f_N(y)) + p(f_N(y), f(y))$$

$$< \epsilon/2 + M_N d(x, y) + \epsilon/2 \leq \epsilon + M d(x, y)$$

So we have that $p(f(x), f(y)) - Md(x,y) < \epsilon$ for all x, y in X. As $\epsilon > 0$ was arbitrary, we have $p(f(x), f(y)) \leq Md(x,y)$. Thus, f is a Lipschitz function.
Now, we use the fact that every 2π-periodic continuous function on $[-\pi, \pi]$ can be approximated uniformly by trigonometric polynomials, i.e. if $f{:}[-\pi,\pi] \to \mathbb{C}$ is a continuous function with $f(-\pi) = f(\pi)$, then there exists a sequence $\{f_n\}$ of trigonometric polynomials that converge uniformly to f. If we prove that every trigonometric polynomial is Lipschitz, then taking any continuous 2π-periodic function $f : [-\pi, \pi] \to \mathbb{C}$ that is not Lipschitz gives us a counterexample. For $k \in \mathbb{Z}$, let $g_k : [-\pi, \pi] \to \mathbb{C}$ be given by $g_k(x) = e^{ikx}$. Then, for $x \neq y$,

$$\frac{|g_k(x) - g_k(y)|}{|x-y|} = \frac{|e^{ikx} - e^{iky}|}{|x-y|} = \frac{2|\sin\frac{k}{2}(x-y)|}{|x-y|} \leq k$$

So g_k is Lipschitz with constant k. Since a finite linear combination of Lipschitz functions is Lipschitz, any trigonometric polynomial is Lipschitz. As an example, we take $f(x) = |x|\ln(|x|)$. If f is Lipschitz, there exists $M > 0$ such that $\frac{|f(x)-f(0)|}{|x-0|} < M$ for all $x \in [-\pi, \pi], x \neq 0$. But $\frac{|f(x)-f(0)|}{|x-0|} = |\ln|x||$ which is unbounded near 0. So f is not Lipschitz.
Note that by observing that polynomials on a bounded interval are Lipschitz, we could also apply Weierstrass approximation theorem to obtain counterexamples.

□

Problem 5. Show that the radius of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} z^{n(n+1)}$$

is 1, and discuss convergence for $z = 1, -1$, and i.

Proof. For this power series, $a_n = \frac{(-1)^m}{m}$ if $n = m(m+1)$ for some $m \in \mathbb{N}$ and 0 otherwise.

$$\limsup |a_n|^{1/n} = \limsup |\frac{(-1)^n}{n}|^{1/n(n+1)} = \limsup \frac{1}{n^{\frac{1}{n(n+1)}}} = \lim \frac{1}{n^{\frac{1}{n(n+1)}}}$$

$$= \frac{1}{\lim n^{\frac{1}{n(n+1)}}} = \frac{1}{e^{\lim \frac{\ln n}{n(n+1)}}} = \frac{1}{e^0} = 1.$$

So $1/R = 1$, hence $R = 1$.
Since $n(n+1)$ is even for all $n \geq 1$, for $z = 1, -1$, the series equals $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = \ln 2$. Let $z = i$. Then the series becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} i^{n(n+1)} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} (-1)^{n(n+1)/2} = \sum_{n=1}^{\infty} \frac{(-1)^{\frac{n(n+3)}{2}}}{n} = 1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - ...$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n(2n-1)}$$

By the alternating series test, this converges. □